Problem 5077. Find all triplets (x, y, z) of real numbers such that

$$\begin{cases} xy(x+y-z)=3 \\ yz(y+z-x)=1 \\ zx(z+x-y)=1 \end{cases}$$

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From the second and third equation it follows that

$$yz(y+z)=zx(z+x) \qquad \Longleftrightarrow \qquad (x-y)(x+y+z)=0$$

If $x+y+z=0$ the first two equations yield $-2xyz=3$ and $-xyz=1$ which is impossible.

If $x=y$ then the system rewrites as

$$\begin{cases} x^2(2x-z)=3 \\ z^2y=1 \\ z^2x=1 \end{cases}$$

Thus $x=\frac{1}{z^2}$ and

$$\frac{1}{z^4}\left(\frac{2}{z^2}-z\right)=3 \qquad \Rightarrow$$

$$3z^6-z^3-2=0 \qquad \Rightarrow \qquad z=-1, \quad z=\sqrt[3]{\frac{2}{3}}$$

Therefore the required triplets are

$$(1,1,-1) \qquad , \qquad \left(\sqrt[3]{\frac{9}{4}}, \sqrt[3]{\frac{9}{4}}, \sqrt[3]{\frac{2}{3}}\right) \qquad \square$$